Math 2040 C Week 13

Spectral Theorem

Let V be a finite dim. inner product space

$T \in L(V)$ has orthonormal eigenbasis

$$\Leftrightarrow \begin{cases} T \text{ is normal} & (\text{if } \mathbb{F} = \mathbb{C}) \\ T \text{ is self-adjoint} & (\text{if } \mathbb{F} = \mathbb{R}) \end{cases}$$

Finite dim vector space $\longleftrightarrow$ $\mathbb{F}^n$

Linear map $\longleftrightarrow$ Matrix

Matrix version of Spectral Theorem ?

We consider the case $\mathbb{F} = \mathbb{R}$ first

Let $V = \mathbb{R}^n$ with standard inner product

$\beta = \{e_1, \cdots, e_n\}$ be the standard basis, $A \in M_{n \times n}(\mathbb{R})$

Consider $T: \mathbb{R}^n \to \mathbb{R}^n$

$$T(x) = Ax \quad , \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Note $M(T, \beta) = A$

β is an orthonormal basis

$\therefore M(T^*, \beta) = A^* = \overline{A}^t = A^t \quad (\mathbb{F} = \mathbb{R})$

i.e. $T^*(x) = A^t x$

Hence, T is self-adjoint $\Leftrightarrow A^t = A$

In that case,

suppose $T \in L(V)$ has orthonormal eigenbasis

$$\alpha = \{w_1, \cdots, w_n\}$$

and $T(w_i) = \lambda_i w_i$

Then $D = M(T,\alpha) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$Q = M(I_V, \alpha, \beta) = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix}$$

α is orthonormal $\Rightarrow$

$$Q^t Q = \begin{bmatrix} — w_1^t — \\ \vdots \\ — w_n^t — \end{bmatrix} \begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \langle w_1, w_1 \rangle & \cdots & \langle w_1, w_n \rangle \\ \vdots & \ddots & \vdots \\ \langle w_n, w_1 \rangle & \cdots & \langle w_n, w_n \rangle \end{bmatrix} = I_n$$

$\therefore$ $Q^{-1} = Q^t$. Also,

$$M(I_V, \beta, \alpha) M(T, \beta) M(I_V, \alpha, \beta) = M(T, \alpha)$$

$$\Rightarrow \quad Q^{-1} A Q = Q^t A Q = D$$

Defn A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called <u>orthogonal</u> if $Q^t Q = I_n$

Spectral Theorem for real symmetric matrix

Let $A \in M_{n\times n}(\mathbb{R})$

A is symmetric $\Leftrightarrow$ $\exists$ orthogonal $Q \in M_{n\times n}(\mathbb{R})$ s.t. $Q^t A Q$ is diagonal

Given symmetric A. How to find Q?

1. Find eigenvalues of A
2. Find basis for each eigenspace and apply Gram-Schmidt Process
3. Resulting vectors from 2 form an orthonormal eigenbasis and the columns of Q

Rmk Eigenvectors of distinct eigenvalues of A are orthogonal.

eg $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 24 \\ 0 & 24 & 7 \end{bmatrix}$

Find orthogonal Q s.t. Q^tAQ is diagonal

Sol ① Find eigenvalues

let $p(t) = \det(A - tI_3)$

$$= \begin{vmatrix} 2-t & 0 & 0 \\ 0 & -7-t & 24 \\ 0 & 24 & 7-t \end{vmatrix}$$

$$= (2-t)\begin{vmatrix} -7-t & 24 \\ 24 & 7-t \end{vmatrix}$$

$$= (2-t)\left[(-7-t)(7-t) - 24^2\right]$$

$$= (2-t)(t^2 - 49 - 24^2)$$

$$= (2-t)(t^2 - 625)$$

$$= -(t-2)(t+25)(t-25)$$

$\Rightarrow$ $\lambda = 2, \pm 25$ are eigenvalues.

② Find orthonormal basis for each eigenspace

For $\lambda = 2$, $A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -9 & 24 \\ 0 & 24 & 5 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -9 & 24 & 0 \\ 0 & 24 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free pivot

x_1 is free variable, let $x_1 = t$

Back substitution $\Rightarrow$ $x_3 = 0$, $x_2 = 0$

$\therefore$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ is a basis for $E(2, A)$

Note $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is unit vector

$\Rightarrow$ $\underbrace{\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}}_{= W_1}$ is an orthonormal basis for $E(2, A)$

Similarly

$$E(25,A): \left\{\begin{bmatrix}0\\3\\4\end{bmatrix}\right\} \xrightarrow{\text{G.S. Process}} \left\{\begin{bmatrix}0\\ \frac{3}{5}\\ \frac{4}{5}\end{bmatrix}\right\}$$

basis $\qquad$ orthonormal basis

($w_2 = \begin{bmatrix}0\\ \frac{3}{5}\\ \frac{4}{5}\end{bmatrix}$)

$$E(-25,A): \left\{\begin{bmatrix}0\\4\\-3\end{bmatrix}\right\} \xrightarrow{\text{G.S. Process}} \left\{\begin{bmatrix}0\\ \frac{4}{5}\\ -\frac{3}{5}\end{bmatrix}\right\}$$

basis $\qquad$ orthonormal basis

($w_3 = \begin{bmatrix}0\\ \frac{4}{5}\\ -\frac{3}{5}\end{bmatrix}$)

③ Find Q

$\{w_1, w_2, w_3\}$ is an orthonormal eigenbasis for A

Let $Q = \begin{bmatrix} | & | & | \\ w_1 & w_2 & w_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

Then Q is orthogonal ($Q^tQ = I_3$)

$Q^tAQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -25 \end{bmatrix}$ is diagonal

eg $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ Find orthogonal Q s.t. Q^tAQ is diagonal

Sol

$p(t) = \det(A - tI_3)$

$= \begin{vmatrix} 3-t & 1 & 1 \\ 1 & 3-t & 1 \\ 1 & 1 & 3-t \end{vmatrix}$

$= (3-t)\begin{vmatrix} 3-t & 1 \\ 1 & 3-t \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 1 & 3-t \end{vmatrix} + (1)\begin{vmatrix} 1 & 3-t \\ 1 & 1 \end{vmatrix}$

$= (3-t)(t^2 - 6t + 8) - (-t+2) + (t-2)$

$= -t^3 + 9t^2 - 24t + 20$

Note $p(2) = 0 \Rightarrow t-2$ is a factor of $p(t)$

By long division, $p(t) = (t-2)(-t^2 + 7t - 10)$

$= -(t-2)^2(t-5)$

$\therefore$ eigenvalues: 2, 5

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $E(2, A)$ is $\left\{ \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{= V_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{= V_2} \right\}$

Apply G.-S. Process to V_1, V_2

$$u_1 = V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad W_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$u_2 = V_2 - \langle V_2, W_1 \rangle W_1 = V_2 - \frac{1}{\sqrt{2}} W_1$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$W_2 = \frac{u_2}{\|u_2\|} = \frac{u_2}{\frac{\sqrt{6}}{2}} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$\{W_1, W_2\}$ is an orthonormal basis for $E(2, A)$

For $\lambda = 5$,

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $E(5, A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Apply G.-S. Process $\rightsquigarrow$ $W_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\{W_3\}$ is an orthonormal basis for $E(5, A)$

Hence, $\{W_1, W_2, W_3\}$ is an eigenbasis for A

Let $Q = \begin{bmatrix} | & | & | \\ W_1 & W_2 & W_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Then Q is orthogonal

$$Q^t A Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

eg $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Find orthogonal Q s.t. Q^tAQ is diagonal

<u>Rmk</u> a trick for finding eigenvalues

① Clearly rank $A = 1 < 3$

$\therefore$ A is not invertible

$\Rightarrow$ 0 is an eigenvalue

② Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of A

$$\begin{aligned} \dim E(0,A) &= \dim \operatorname{null} A \\ &= 3 - \operatorname{rank} A \\ &= 2 \end{aligned}$$

$\therefore$ 0 has multiplicity ≥ 2

Take $\lambda_1 = \lambda_2 = 0$.

Fact: Sum of eigenvalues

= trace A (Sum of diagonal entries)

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 1 + 4 + 1 = 6$

$\Rightarrow \lambda_3 = 6$

One correct answer of Q

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

with $Q^tAQ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

A simple application

$$3(x^2+y^2+z^2)+2(xy+yz+zx)=2$$

Note
$$[x\ y\ z]\underbrace{\begin{bmatrix}3&1&1\\1&3&1\\1&1&3\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix}=2$$

Previous calculation:

$$Q^tAQ=D, \text{ where } D=\begin{bmatrix}2&0&0\\0&2&0\\0&0&5\end{bmatrix}$$

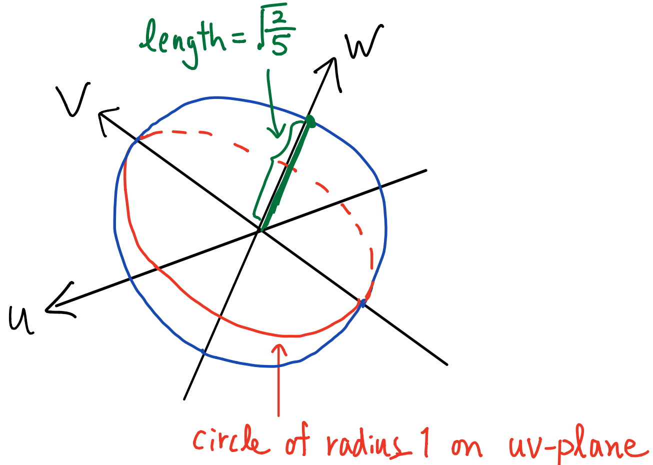
$$Q=\begin{bmatrix}-\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&\frac{2}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{bmatrix} \text{ is orthogonal}$$

Let
$$\begin{bmatrix}u\\v\\w\end{bmatrix}=Q^t\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{1}{\sqrt2}x+\frac{1}{\sqrt2}z\\-\frac{1}{\sqrt6}x+\frac{2}{\sqrt6}y-\frac{1}{\sqrt6}z\\\frac{1}{\sqrt3}x+\frac{1}{\sqrt3}y+\frac{1}{\sqrt3}z\end{bmatrix}$$

(an orthogonal change of coordinates)

Then
$$[u\ v\ w]D\begin{bmatrix}u\\v\\w\end{bmatrix}=[x\ y\ z]QDQ^t\begin{bmatrix}x\\y\\z\end{bmatrix}$$
$$=[x\ y\ z]A\begin{bmatrix}x\\y\\z\end{bmatrix}$$
$$=2$$

i.e. $2u^2+2v^2+5w^2=2 \rightsquigarrow$ Ellipsoid



Rmk w-axis is span$\{(1,1,1)\}$

uv-plane is the plane $x+y+z=1$

Determinant for operators

Let V be a vector space. $\dim V = n < \infty$

α be an ordered basis.

$$\boxed{\begin{array}{l}\underline{\text{Defn}}\ \text{Let } T \in \mathcal{L}(V). \text{ Define} \\ \text{the } \underline{\text{characteristic polynomial}} \text{ of } T \\ \text{to be } p(t) = \det(M(T,\alpha) - tI_n)\end{array}}$$

Q: Well-defined? Indept of choice of β?

Suppose α, β are ordered basis of V

Let $A = M(T,\alpha)$, $B = M(T,\beta)$

$Q = M(I_V, \alpha, \beta)$

Then $Q^{-1}BQ = A$

$$\begin{aligned}\det(A - tI_n) &= \det(Q^{-1}BQ - tI_n) \\ &= \det(Q^{-1}BQ - Q^{-1}(tI_n)Q) \\ &= \det[Q^{-1}(B - tI_n)Q] \\ &= \det Q^{-1} \cdot \det(B - tI) \cdot \det Q \\ &= \frac{1}{\det Q} \cdot \det(B - tI) \cdot \det Q \\ &= \det(B - tI)\end{aligned}$$

$\therefore$ $\boxed{\text{Char. poly. of } T \text{ is well-defined}}$

$$\boxed{\begin{array}{l}\underline{\text{Defn}}\ \text{Let } \lambda \text{ be an eigenvalue of } T \in \mathcal{L}(V) \\ p(t) \text{ be the char. polynomial of } T. \\ \text{Then } p(\lambda) = 0. \text{ Define} \\ \text{algebraic multiplicity of } \lambda = \text{multiplicity of } \lambda \text{ in } p(t) \\ \text{geometric multiplicity of } \lambda = \dim E(\lambda, T)\end{array}}$$

Rmk Similar definitions for square matrices

> **Prop** algebraic multiplicity $\geq$ geometric multiplicity

> **Defn** Let $p(t) \in P(\mathbb{F})$ be a polynomial
>
> $p(t)$ is said to <u>split</u> over $\mathbb{F}$ if
>
> $$p(t) = c(t-a_1)(t-a_2)\cdots(t-a_n)$$
>
> for some $c, a_1, a_2, \cdots, a_n \in \mathbb{F}$

eg

$$t^3 - 1 = (t-1)\underbrace{(t^2+t+1)}^{\text{irreducible over } \mathbb{R}}$$

$$= (t-1)(t-e^{\frac{2\pi i}{3}})(t-e^{-\frac{2\pi i}{3}})$$

$\therefore$ t^3-1 splits over $\mathbb{C}$, not over $\mathbb{R}$

> **Fundamental Theorem of Algebra**
>
> Every complex polynomial splits over $\mathbb{C}$

Conditions for diagonalizable

eg1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbb{F} = \mathbb{R}$$

char poly. $p(t) = -(t-1)(t^2+1)$

(does not split over $\mathbb{R}$)

only one eigenvalue 1, $E(1, A) \neq \mathbb{R}^3$

$\Rightarrow$ A is not diagonalizable over $\mathbb{R}$

Rmk If $\mathbb{F} = \mathbb{C}$,

$p(t) = -(t-1)(t-i)(t+i)$ splits

3 distinct eigenvalues

$\Rightarrow$ A is diagonalizable over $\mathbb{C}$

eg 2

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -2 & -1 \\ 0 & 3 & -1 \\ 4 & -4 & 3 \end{bmatrix}$$

Both have char poly. $-(t-5)(t-3)^2$

$\Rightarrow$ Both have eigenvalues $5, 3$

- $\dim E(5,A) = \dim E(5,B) = 1$
- $A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \dim E(3,A) = 2$

$B - 3I = \begin{bmatrix} 2 & -2 & -1 \\ 0 & 0 & -1 \\ 4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \dim E(3,B) = 1$

$\dim E(5,A) + \dim E(3,A) = 3$

$\Rightarrow$ A is diagonalizable

$\dim E(5,B) + \dim E(3,B) = \underbrace{2 < 3}$

Not enough lin indept e.vector to form basis

$\Rightarrow$ B is not diagonalizable

Thm Let V be a vector space, $\dim V < \infty$.

Then $T \in L(V)$ is diagonalizable

$\Leftrightarrow$ Characteristic polynomial of T splits,

and for each eigenvalue

algebraic multiplicity = geometric multiplicity

Jordan Canonical form

A Jordan block is a square matrix of the form

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \text{\huge 0} \\ & & \lambda & 1 & \\ & & & \ddots & \ddots \\ & \text{\huge 0} & & & \ddots & 1 \\ & & & & & \lambda \end{bmatrix}$$

λ on diagonal

1 on "superdiagonal"

0 for other entries

A square matrix is said to be in Jordan canonical form if it has the following form

$$\begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix}$$

where each J_i is a Jordan block

eg

$$\left[\begin{array}{cccccc} \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & \\ 0 & & \boxed{7} & \end{array}\right]$$

3 Jordan blocks of sizes 3, 2, 1

6×6

Rmk A $n \times n$ diagonal matrix is in Jordan form, consisting of Jordan block of size 1

> **Thm** Let V be finite dim. vector space. Suppose $T \in \mathcal{L}(V)$ and its char poly splits. Then $\exists$ a basis β of V s.t. $M(T, \beta)$ is in Jordan canonical form

Thm (Matrix version)

Let $A \in M_{n\times n}(\mathbb{F})$ and its char poly splits

Then $\exists$ an invertible $Q \in M_{n\times n}(\mathbb{F})$ s.t.

$Q^{-1}AQ$ is in Jordan canonical form

eg

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad 5\times 5$$

char poly $p(t) = -t^4(t-1)$ splits

$E(0,A)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\}$

$E(1,A)$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim E(0,A) + \dim E(1,A) = 3 < 5$

$\Rightarrow$ A is not diagonalizable

A has no eigenbasis but

a Jordan Canonical basis $\{v_1, v_2, v_3, v_4, v_5\}$

where

$$\text{Let } Q = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \parallel & \parallel & \parallel & \parallel & \parallel \\ v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \end{array}$$

$$Q^{-1}AQ = \begin{bmatrix} \boxed{1} & & O \\ & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \\ O & & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{bmatrix}$$

Summary for linear operators

Vector Space ($\dim V = n < \infty$)

Any field $\mathbb{F}$

n distinct eigenvalues $\Rightarrow$ Diagonalizable

$\bigoplus E(T, \lambda_i) = V$ $\Longleftrightarrow$ Diagonalizable

Diagonalizable $\Updownarrow$ Eigenbasis exists

- Char. poly splits
- For each eigenvalue geom. mult = alg. mult.

$\Longleftrightarrow$ Diagonalizable / Eigenbasis exists

Diagonalizable / Eigenbasis exists $\Downarrow$ Char. poly splits

Jordan Canonical form exists $\Longleftrightarrow$ Char. poly splits

Inner Product Space

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Orthonormal eigenbasis exists $\Rightarrow$ Diagonalizable / Eigenbasis exists

Self-adjoint $T^* = T$ $\xLeftrightarrow{\text{if } \mathbb{F} = \mathbb{R}}$ Orthonormal eigenbasis exists

Self-adjoint $T^* = T$ $\Downarrow$ Normal $T^*T = TT^*$

Normal $T^*T = TT^*$ $\xLeftrightarrow{\text{if } \mathbb{F} = \mathbb{C}}$ Orthonormal eigenbasis exists

Char. poly splits $\xLeftrightarrow{\text{Schur}}$ $\exists$ orthonormal basis β s.t. $M(T, \beta)$ is upper triangular